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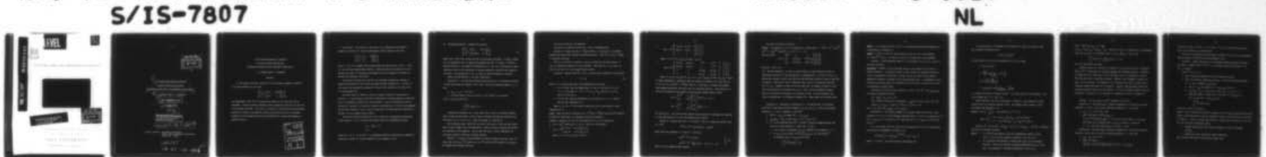

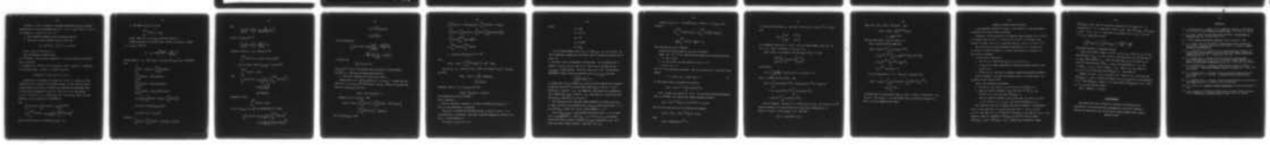

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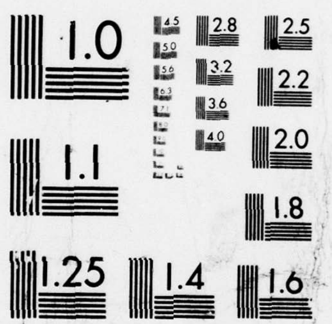

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ON THE UNIFORM ASYMPTOTIC STABILITY OF CERTAIN LINEAR TIME-VARYING DIFFERENTIAL EQUATIONS WITH UNBOUNDED COEFFICIENTS

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On the Uniform Asymptotic Stability

of Certain Linear Time-Varying

Differential Equations with Unbounded Coefficients

A. P. Morgan and K. S. Narendra

Abstract

In this paper sufficient conditions for the uniform asymptotic stability of certain equations of the form

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A(t) & -B^T(t) \\ C(t) & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

are established, where A(t) is bounded and stable but B(t) and C(t) may be unbounded. The key condition implying uniform asymptotic stability is that B(t) be "uniformly exciting". This result is related to previous work characterizing stability in terms of excitedness conditions when B(t) and C(t) are bounded. Interest in the unbounded case has been stimulated by recent developments in adaptive control.

I. Introduction: The purpose of this paper is to characterize the uniform asymptotic stability of certain nonautonomous linear systems of the form

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A & -B^T \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

where $A = A(t)$ is a bounded time varying stable nxn matrix and $B = B(t)$, $C = C(t)$ are time varying unbounded mxn matrices. Questions about the stability of such systems has arisen in connection with adaptive control schemes as described in Narendra and Valavani [7],[8],[9].

The special case that $B(t)$ and $C(t)$ are uniformly bounded was treated in Morgan and Narendra [6] and related nonlinear systems in Morgan [4]. We show in this paper that appropriate extensions of some of the results in [6] carry over to the unbounded case.

The main result is Theorem 4, which is stated in Section 3 and proven in Section 4. It gives general sufficient conditions for uniform asymptotic stability. Rate of convergence estimates are also provided. By way of introduction and exposition, in Section II we discuss a special case (Theorem 2, corollary and examples). We also show (Theorem 3) that a class of equations not satisfying the conditions of Theorem 2 are not asymptotically stable.

Some previous work has been done on the stability of unbounded nonautonomous linear systems. In [1], Artstein and Infante study the 2-dimensional system

$$\dot{x} = - a(t)x - by$$
$$\dot{y} = x$$

where $a(t) \geq 0$, $b > 0$, and $a(t) \to \infty$, obtaining sufficient conditions for asymptotic stability in terms of a growth condition on the integral of $a(t)$.

II. Preliminary Results: Consider the equation

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A(t) & -B^T(t) \\ B(t)P(t) & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{1}$$

where $A(t)$ is nxn, $B(t)$ is mxn and both are piecewise continuous. Further, assume $A(t)$ is bounded and $P(t)$ is smooth and bounded with $\dot{P}(t) + P(t)A(t) + A^T(t)P(t)$ negative definite. See Morgan and Narendra [6], p. 166, for a discussion of these conditions and for the definitions of uniform asymptotic stability and asymptotic stability. The abbreviation u.a.s. is used for uniform asymptotic stability.

Before stating Theorem 1, we need a definition.

Definition 1: The uniformly bounded mxn matrix $B_0(t)$ is "uniformly exciting" if there exist positive numbers T and ε and a strictly increasing sequence $s_i \to \infty$ with

1. $s_{i+1} - s_i \leq T$ for all i
2. for each index i and unit vector $y_0 \in R^m$, there is an interval $(a,b) \subseteq (s_i, s_{i+1})$ such that

$$\left| \int_a^b B^T(\tau) y_0 d\tau \right| \geq \varepsilon.$$

In Morgan and Narendra [6], it was shown that if $B(t)$ is uniformly bounded, then a condition equivalent to $B(t)$'s being uniformly exciting was necessary and sufficient for (1) to be u.a.s. (See also Morgan [4], Theorem 6).

The following result, which will be used in the proof of the corollary below, gives further insight into this definition. Almost periodic functions are defined and discussed in the appendix. Let us note here that a linear combination of continuous periodic functions is almost periodic.

Theorem 1: If $B_0(t)$ is an mxn matrix of almost periodic functions, and, for every unit vector $y_0 \in R^m$, there is a $t_0 \in R^+$ such that $|B^T(t_0)y_0| \neq 0$, then $B_0(t)$ is bounded and uniformly exciting.

The proof is given in the appendix.

Now we have a stability result for a class of unbounded $B(t)$.

Theorem 2: Assume that $B(t) = e^{\alpha t}B_0(t)$ where $\alpha \geq 0$ and $B_0(t)$ is uniformly bounded and smooth with uniformly bounded derivative. Then (1) is u.a.s. if $B_0(t)$ is uniformly exciting.

This is a corollary to Theorem 5, which is stated in the next section. A rate of convergence estimate is also available. Note that if $\alpha < 0$, then it is easy to see that (1) is not asymptotically stable.

Corollary: Suppose that $B(t) = b(t)$ is an $m \times 1$ matrix defined by the equation

$$\dot{b} = Cb \tag{2}$$

where C is a constant $m \times m$ matrix with eigenvalues as follows:

1. if m is even, then the eigenvalues of C are of the form $\alpha \pm i\beta_j$ for $j = 1,\ldots,m/2$ where $i = \sqrt{-1}$, $\alpha \geq 0$, $\beta_{j_1} \not\equiv \beta_{j_2}$ (mod 2π) if $j_1 \neq j_2$, and $\beta_j \neq 0$ for all j.
2. If m is odd, then the eigenvalues are as above for $j = 1,\ldots,(m-1)/2$ and the real eigenvalue is α.

Then (1) is u.a.s. if the initial conditions for (2) are nonzero on each eigenspace of C.

<u>Proof</u>: Under the above conditions $b(t) = e^{Ct}b(0) = e^{\alpha t}B_0(t)$ has the form required by Theorem 2 where $B_0(t)$ is uniformly exciting by Theorem 1.

Examples: Equation (1) with the following $b(t)$ is u.a.s. by Theorem 2.

1. $b(t) = k_0 e^{\alpha t}$ where $k_0 \neq 0$, $\alpha \geq 0$. ($b(t)$ is a 1×1 matrix).
2. $$b(t) = e^{\alpha t}\begin{bmatrix} \cos \beta t & \sin \beta t \\ -\sin \beta t & \cos \beta t \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$
   where $\alpha \geq 0$, $\beta \neq 0$, $(k_1,k_2) \neq (0,0)$.

3.
$$b(t) = e^{\alpha t}\begin{bmatrix} \cos\beta t & \sin\beta t & 0 \\ -\sin\beta t & \cos\beta t & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

where $\alpha \geq 0$, $\beta \neq 0$, $(k_1,k_2) \neq (0,0)$, $k_3 \neq 0$.

4.
$$b(t) = e^{\alpha t}\begin{bmatrix} \cos\beta_1 t & \sin\beta_1 t & e^{-\alpha t} & te^{-\alpha t} \\ -\sin\beta_1 t & \cos\beta_1 t & 0 & e^{-\alpha t} \\ 0 & 0 & \cos\beta_2 t & \sin\beta_2 t \\ 0 & 0 & -\sin\beta_2 t & \cos\beta_2 t \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix}$$

where $\alpha \geq 0$, $\beta_1 \not\equiv \beta_2 \pmod{2\pi}$, $\beta_1 \neq 0$, $\beta_2 \neq 0$, $(k_1,k_2) \neq (0,0)$, $(k_3,k_4) \neq (0,0)$.

It seems reasonable to conjecture that the converse of Theorem 2 is true; that is, if (1) is u.a.s. and $B(t) = e^{\alpha t}B_0(t)$ where $\alpha \geq 0$ and $B_0(t)$ is smooth and bounded with bounded derivative, then $B_0(t)$ must be uniformly exciting. Thus equation (1) with the following 2x1 $B(t)$ would not be uniformly asymptotically stable:

$$B(t) = \begin{bmatrix} e^{\alpha_1 t} & 0 \\ 0 & e^{\alpha_2 t} \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \text{ where } \alpha_1 > \alpha_2 > 0$$

$$B(t) = \begin{bmatrix} e^{\alpha t} & 1 \\ 0 & e^{\alpha t} \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \text{ where } \alpha > 0$$

for any (k_1,k_2). The result below supports this conjecture by showing that equation (1) with many such $B(t)$ is not even asymptotically stable.

Theorem 3: Suppose that $A(t) = -1$ and $B(t) = b(t)$ is a smooth mxl matrix with $m \geq 2$ and

$$b^T(t) = (e^t + b_1(t), b_2(t), \ldots, b_m(t))$$

where there are constants $\varepsilon > 0$ and $k > 0$ such that

$$\left.\begin{aligned} ke^{(\frac{1}{2} - \varepsilon)t} &\geq |b_j(t)| \\ ke^{(\frac{1}{2} - \varepsilon)t} &\geq \left|\frac{d}{dt} b_j(t)\right| \quad \text{for } j = 1,\ldots,m. \end{aligned}\right\} \quad (3)$$

Then (1) is not asymptotically stable.

The proof is given in Section 4.

Example: The equation (1) is not asymptotically stable when 1. $b^T(t) = [e^t + k_1 e^{\alpha_1 t}, k_2 e^{\alpha_2 t}, \ldots, k_m e^{\alpha_m t}]$ where $\frac{1}{2} > \alpha_j$ for $j = 1, \ldots, m$.

and also when 2.

$$b(t) = \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{\alpha t}\cos \beta t & e^{\alpha t}\sin \beta t \\ 0 & -e^{\alpha t}\sin \beta t & e^{\alpha t}\cos \beta t \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

where $\frac{1}{2} > \alpha$ and $\beta \neq 0$.

III. The Main Theorems: To prove our main results, we must extend the idea of "uniformly exciting" in a way appropriate to unbounded $B(t)$. We will, in fact, redefine this concept in such a way that the new definition reduces to the old when $B(t)$ is bounded. Then we will show in Theorem 4 that, if $B(t)$ is uniformly exciting, (1) is u.a.s. (However, unlike in the case that $B(t)$ is bounded, we have no proof that this condition is necessary for u.a.s.) Also, we will establish an upper bound on the speed of convergence of solutions to zero. The key concept is the "rate of persistence", as discussed in Morgan [4]. These ideas are outlined below.

Definition 2: (Extension of Definition 1). The matrix $B(t)$ is "uniformly exciting" if there exist positive numbers T_0, T_1, T_2, and ε_0, a strictly increasing sequence $s_i \to \infty$, and a sequence u_i such that

1. $|B(\tau)| \leq u_{i+1}$ for almost all $\tau \in (s_i, s_{i+1})$
2. $u_{i+1}(s_{i+1} - s_i) \leq T_0$ and $(s_{i+1} - s_i) \leq T_0$ for all i
3. if I is an interval with $\mu(I) \geq T_1$ where μ denotes Lebesgue measure then there is an open subset $S \subseteq I$ with $\mu(S) \geq T_2$ and
   a. S is a union of intervals (s_i, s_{i+1}) defined by the sequence $s_i \to \infty$
   b. if y_0 is a unit vector in R^m and $(s_i, s_{i+1}) \subseteq S$, then there is an interval $(a,b) \subseteq (s_i, s_{i+1})$ such that

$$\left| \int_a^b B^T(\tau) y_0 \, d\tau \right| \geq \varepsilon_0.$$

Remark: It is easily seen that if B(t) is a bounded mxn matrix then Definition 1 is equivalent to Definition 2.

Remark: If $|B(\tau)| \leq ke^{\alpha\tau}$ for some positive k and α for all τ then letting $s_i = \ell n((i)^{1/\alpha})$ and $u_i = ki$ parts (1) and (2) of Definition 2 are satisfied.

Theorem 4: The equilibrium state x = 0 of equation (1) is u.a.s. if B(t) is uniformly exciting.

Discussion: Theorem 4 is the principal result of this paper and its proof is given in Section 4. We discuss briefly in what follows the important concept of "rate of persistence" which plays a crucial role in the proof of the uniform asymptotic stability of systems of the type described here. A consequence of this is that we can derive rates of convergence for (1) directly in terms of the rate of persistence as shown below.

From now on, we will use the notation $z = (x,y) \in R^n \times R^m$. Let a_0, w_0, p_1, p_2 be positive constants such that

(i) $|A(t)| \leq a_0$ for $t \in R^+$

(ii) $x^T(\dot{P}(t) + A^T(t)P(t) + P(t)A(t))x \leq -w_0|x|^2$ for all $(t,x) \in R^+ \times R^n$

(iii) $p_1 \geq 1$, $p_2 \leq 1$, and $p_2^{\,2} \leq |P(t)| \leq p_1^{\,2}$ for $t \in R^+$

Let $A = \{z \in R^n \times R^m \mid 2p_1/p_2 \geq |z| \geq p_2/p_1\}$.

Definition 3: The positive number r is an upper bound for the rate of persistence of (1) in A if the following condition holds. If z(t) is a solution to (1) for $t \in [t_0, t_0 + r]$ for some $t_0 \in R^+$, then there is a $t_1 \in [t_0, t_0 + r]$ such that $z(t_1) \notin A$.

If r is an upper bound for the rate of persistence of (1) in A, then letting Z(t) denote the fundamental solution of (1)

$$|Z(t)Z(t_0)^{-1}| \leq (2p_1/p_2)e^{-L(t-t_0)} \quad \text{for } t \geq t_0 \geq 0$$

where L = ln (2)/r. (For more details, see Morgan [4]).

If T_0, T_1, T_2 and ε_0 are defined as in Definition 2, then for system (1) with $B(t)$ uniformly exciting we have

$$r = 3T[4\, p_1^{\,4}/p_2^{\,2}\phi]^{*}$$

an upper bound for the rate of persistence of (1) in A where

$$T = \max\,\{T_0, T_1/3\}$$

$$\phi = \frac{w_0\gamma^2}{16}\min\left\{\frac{1}{a_0^{\,2}T_0}\,,\ \Delta_0^{\,2}\ T_2\right\}$$

$$\Delta_0 = \min\left\{\frac{1}{2},\ \frac{p_2\gamma}{8p_1T_0}\right\}$$

$$\gamma = \varepsilon_0(p_2/2p_1)\ \min\left\{\frac{1}{(a_0+T_0)T_0}\ ,\ \frac{1}{2}\right\}\ \text{and}$$

[*] denotes the greater integer function. (In other words, for real number p, [p] is an integer and $p + 1 \geq [p] > p$.)

Since Theorems 2 and 5 are corollaries to Theorem 4, the examples for these results serve to illustrate Theorem 4 also. The example below, however, cannot be derived from Theorems 2 or 5.

Example: Let $s_i = \sum_{n=1}^{i} \frac{1}{n}$. Then define $b_0\colon R^+ \to R^1$ by

$$b_0(t) = \begin{cases} 1 & \text{if } t \in [s_i, s_i + \frac{1}{2}(s_{i+1} - s_i)) \text{ for some } i \\ -1 & \text{if } t \in [s_i + \frac{1}{2}(s_{i+1} - s_i),\ s_{i+1}) \text{ for some } i. \end{cases}$$

Define h: $R^+ \to R^+$ by $h(t) = 2/(s_{i+1} - s_i)$ if $t \in [s_i, s_{i+1})$. Let $b(t) = h(t)b_0(t)$. Then $b(t)$ is uniformly exciting.

Remark: 1. It follows from Theorem 4 that the 2-dimensional system (1) with $A(t) = -a$ for positive constant a and $B(t) = b(t)$ is u.a.s.

2. For comparison with Theorem 5, observe that $b_0(t)$ is not uniformly exciting. (This $b_0(t)$ appeared in Morgan and Narendra [6], p. 16 as part of an example of a bounded system which is not u.a.s.)

Proof: Note that $s_{i+1} - s_i = \frac{1}{i+1}$.

Let $T_0 = 2$ and $u_{i+1} = 2/(s_{i+1} - s_i)$. Then parts 1 and 2 of Definition 2 are immediate. Given any (s_i, s_{i+1}), if we define $(a,b) = (s_i, s_i + \frac{1}{2}(s_{i+1} - s_i))$, then

$$\left| \int_a^b b(\tau)d\tau \right| = \frac{2}{s_{i+1}-s_i}(b-a) = 1 = \varepsilon_0.$$

Part 3 of Definition 2 follows.

Theorem 5 below is a corollary to Theorem 4 for the case that $B(t) = h(t)B_0(t)$ where $h: R^+ \to R^1$ and $B_0(t)$ is uniformly bounded and uniformly exciting. Basically, we assume that $h(t)$ obeys parts 1 and 2 and $B_0(t)$ part 3 of Definition 2, and we show then that $B(t)$ is uniformly exciting. However, some "tameness" condition on $B_0(t)$ is also required. For example, it is sufficient that $B_0(t)$ be smooth with uniformly bounded derivative as in Theorem 2. However, many piecewise smooth and piecewise Lipschitz $B_0(t)$ will also suffice. We call the needed condition on $B_0(t)$ assumption E, because it is an extension of assumptions D and D' in Morgan [4].

Notation. Let $(w)_k$ denote the k^{th} component of vector w.

Definition 4: The matrix $B_0(t)$ satisfies assumption E if there are positive constants γ, δ, and L, and strictly increasing sequence $t_i \to \infty$ such that

1. $t_{i+1} - t_i \geq L$ for all i, and
2. given the index i, $(B^T(t_*)y_0)_k \geq \gamma$

   if $y_0 \in R^m$ and $(B_0{}^T(t_*)y_0)_k \geq \gamma$ for some k and for some $t_* \in (t_i, t_{i+1})$, then there is an interval $(a,b) \subseteq (t_i, t_{i+1})$ with $t_* \in (a,b)$ and $b - a \geq \delta$ such that $(B_0{}^T(t)y_0)_k \geq \gamma/2$ for $t \in (a,b)$.

Proposition: The matrix $B_0(t)$ satisfies assumption E if there are positive constants L and b_0 and a strictly increasing sequence $t_i \to \infty$ such that

1. $t_{i+1} - t_i \geq L$ for all i
2. Given the index i,

   $\frac{d}{dt} B_0(t)$ exists and $\left| \frac{d}{dt} B_0(t) \right| \leq b_0$ for $t \in (t_i, t_{i+1})$.

The proof is easy. For any $\gamma > 0$, we have $\delta = \gamma/2\, b_0$. Note that the value of $B_0(t)$ at the points t_i is unrestricted.

The role of assumption E in this paper is similar to the role of the piecewise smooth category PS defined by Yuan and Wonham in [11].

Theorem 5: Suppose that $B_0(t)$ is a uniformly bounded piecewise continuous mxn matrix. Suppose that $h: R^+ \to R^1$ is a piecewise continuous function. Let $B(t) = h(t)B_0(t)$.

Then (1) is u.a.s. if

1. $B_0(t)$ satisfies assumption E and is uniformly exciting
2. there are positive constants T_* and h_*, a strictly increasing sequence $s_i \to \infty$, and a sequence v_i such that
   a. $|h(\tau)| \leq v_{i+1}$ for almost all $\tau \in (s_i,s_{i+1})$ for all i
   b. $v_{i+1}(s_{i+1} - s_i) \leq T_*$ for all i, and $s_{i+1} - s_i \to 0$ as $i \to \infty$
   c. given the index i, either $h(t) \geq 0$ for all $t \in (s_i,s_{i+1})$ or $h(t) \leq 0$ for all $t \in (s_i,s_{i+1})$, and

$$\int_{s_i}^{s_{i+1}} |h(\tau)|\,d\tau \geq h_*.$$

The proof is given in Section 4.

Remark: Rate of persistence bounds for Theorem 5 can be easily derived from the discussion following Theorem 4 and the parameters listed in the first part of the proof of Theorem 5. Theorem 5 divides the problem of finding B(t) for which (1) is u.a.s. into two completely independent parts:

1. finding bounded $B_0(t)$ that satisfy assumption E and are uniformly exciting
2. finding h(t) that satisfy the listed conditions.

The corollary below gives a particularly simple case.

Corollary: If $B(t) = h(t)B_0(t)$ is piecewise continuous with $B_0(t)$ uniformly bounded and $h(t)$ strictly increasing with $h(t) \to \infty$ as $t \to \infty$ and $h^{-1}(i+1) - h^{-1}(i) \to 0$ for integers $i \to \infty$, then (1) is u.a.s. if

1. $B_0(t)$ satisfies assumption E and is uniformly exciting
2. there exist positive constants T_* and h_* such that

$$T_* \geq i(h^{-1}(i+1) - h^{-1}(i)) \geq h_* \text{ for all } i.$$

Proof: Let $s_i = h^{-1}(i)$ and $v_i = i$.

Note that Theorem 2 follows at once.

IV. The Proofs: In this section, Theorems 4, 5, and 3 are proven, in that order.

Proof of Theorem 4:

1. Letting $V(t,z) = x^TP(t)x + y^Ty$, we see that $-\dot{V}(t,z) \geq w_0|x|^2 \geq 0$. Therefore, 0 is uniformly stable. In fact, it is easy to see that

$$|Z(t)Z(t_0)^{-1}| \leq p_1/p_2 \text{ for all } t \geq t_0 \geq 0.$$

Suppose $z(t)$ is a solution to (1) with $z(t) \in A$ for $t \in I$, where I is some interval with $\mu(I) = 3qT$ and q is a positive integer. We will see that there is an upper bound on the size of q, and this will establish the stated upper bound r for the rate of persistence of (1) in A. The u.a.s. of (1) then follows.

2. Let $I = \cup I_k$ for $k = 1,\ldots,q$ where $\mu(I_k) = 3T$. For each I_k, there is an open subset $S_k \subseteq I_k$ as in Condition 4 of Definition 2.

Then

$$\int_{I_k} -\dot{V}(\tau,z(\tau))d\tau \geq \int_{S_k} -\dot{V}(\tau,z(\tau))d\tau \geq w_0 \int_{S_k} |x(\tau)|^2 d\tau =$$

$$w_0 \sum_i \int_{s_i}^{s_{i+1}} |x(\tau)|^2 d\tau \geq w_0 \sum_i \frac{1}{s_{i+1}-s_i} \left(\int_{s_i}^{s_{i+1}} |x(\tau)| d\tau \right)^2$$

where the sum Σ is over all intervals $(s_i, s_{i+1}) \subseteq S_k$.

3. Now suppose $(s_i, s_{i+1}) \subseteq S_k$ and

$$\int_{s_i}^{s_{i+1}} |x(\tau)|d\tau \lesssim \gamma/4a_0.$$

Claim: There is a $t_0 \in (s_i, s_{i+1})$ such that $|x(t_0)| \geq \gamma$.

We defer the proof of the claim to part 5 below. Fix this t_0. Choose $t_1 \in (s_i, s_{i+1})$ such that

$$|t_1 - t_0| = \min\left\{\frac{s_{i+1}-s_i}{2}, \frac{p_2\gamma}{8p_1u_{i+1}}\right\}$$

We may assume $t_1 > t_0$. Then $(t_0, t_1) \leq (s_i, s_{i+1})$ and $u_{i+1}(t_1-t_0) \leq \gamma/8(p_1/p_2)$.

Now

$$\int_{t_0}^{t_1} |x(\tau) - x(t_0)|d\tau \leq \int_{t_0}^{t_1}\int_{t_0}^{\tau} |\dot{x}(\sigma)|d\sigma d\tau \leq$$

$$\int_{t_0}^{t_1}\int_{t_0}^{\tau} |A(\sigma)x(\sigma) - B^T(\sigma)y(\sigma)|d\sigma d\tau \leq$$

$$\int_{t_0}^{t_1}\int_{t_0}^{\tau} (a_0|x(\sigma)| + |B(\sigma)||y(\sigma)))d\sigma d\tau \leq$$

$$\int_{t_0}^{t_1}\int_{t_0}^{t_1} (a_0|x(\sigma)| + 2(p_1/p_2)|B(\sigma)|)d\sigma d\tau \leq$$

$$(t_1-t_0)\left(a_0\int_{t_0}^{t_1} |x(\sigma)|d\sigma + 2(p_1/p_2)\int_{t_0}^{t_1} |B(\sigma)|d\sigma\right) \leq$$

$$(t_1-t_0)\left(\frac{\gamma}{4} + 2(p_1/p_2)u_{i+1}(t_1-t_0)\right) \leq$$

$$(t_1-t_0)\left(\frac{\gamma}{4} + \frac{\gamma}{4}\right) = (t_1-t_0)\frac{\gamma}{2}.$$

Therefore,

$$\int_{t_0}^{t} |x(\tau)|d\tau \geq \int_{t_0}^{t_1} |x(t_0)|d\tau - (t_1-t_0)\frac{\gamma}{2} \geq (t_1-t_0)\frac{\gamma}{2}.$$

Now,

$$\frac{t_1 - t_0}{s_{i+1}-s_i} = \min\left\{\frac{1}{2}\ ,\ \frac{p_2\gamma}{8p_1u_{i+1}(s_{i+1}-s_i)}\right\}.$$

But $T_0 \geq u_{i+1}(s_{i+1}-s_i)$, so

$$\frac{t_1 - t_0}{s_{i+1}-s_i} \geq \min\left\{\frac{1}{2}\ ,\ \frac{p_2\gamma}{8p_1T_0}\right\} = \Delta_0 .$$

Therefore, we have $t_1 - t_0 \geq \Delta_0(s_{i+1}-s_i)$ and

$$\int_{s_i}^{s_{i+1}} |x(\tau)|d\tau \geq (t_1-t_0)\,\frac{\gamma}{2} \geq (s_{i+1}-s_i)\Delta_0\,\frac{\gamma}{2}$$

4. Now, if there is some $(s_{i_0},s_{i_0+1}) \subseteq S_k$ such that

$$\int_{s_{i_0}}^{s_{i_0+1}} |x(\tau)|d\tau \geq \gamma/4a_0 ,$$

then

$$\int_{I_k} -\dot{V}(\tau,z(\tau))d\tau \geq w_0\,\frac{1}{s_{i_0+1}-s_{i_0}}\left(\int_{s_{i_0}}^{s_{i_0+1}} |x(\tau)|d\tau\right)^2 \geq$$

$$\geq w_0\,\frac{1}{T_0}\left(\frac{\gamma}{4a_0}\right)^2$$

$$= w_0\gamma^2/16a_0^{\,2}T_0.$$

Otherwise, we have

$$\int_{s_i}^{s_{i+1}} |x(\tau)|d\tau \leq \gamma/4a_0$$

for all $(s_i,s_{i+1}) \subseteq S_k$, and, by the material in 3. above,

$$\int_{I_k} -\dot{V}(\tau,z(\tau))d\tau \geq w_0 \sum_i \frac{1}{s_{i+1}-s_i}\left(\int_{s_i}^{s_{i+1}} |x(\tau)|d\tau\right)^2$$

$$\geq w_0 \sum_i \frac{1}{(s_{i+1}-s_i)}\left((s_{i+1}-s_i)\Delta_0\,\frac{\gamma}{4}\right)^2$$

$$= w_0 \Delta_0^2 \frac{\gamma^2}{16} \sum_i (s_{i+1}-s_i)$$

$$\geq w_0 \Delta_0^2 \frac{\gamma^2}{16} T_2 .$$

We can conclude that

$$\int_{I_k} - \dot{V}(\tau,z(\tau))d\tau \geq \min\left\{\frac{w_0\gamma^2}{16a_0^2T_0}, \frac{w_0\gamma^2\Delta_0^2T_2}{16}\right\}$$

$$= \frac{w_0\gamma^2}{16} \min\left\{\frac{1}{a_0^2T_0}, \Delta_0^2T_2\right\} = \phi.$$

It follows that

$$- \int_I \dot{V}(\tau,z(\tau))d\tau \geq q\phi.$$

Since $p_1^2|z|^2 \geq V(t,z)$, this implies that $p_1^2(2p_1/p_2)^2 \geq q\phi$ and therefore $4p_1^4/p_2^2\phi \geq q$. This puts the needed upper bound on q.

5. Now we will establish the claim stated in Part 3.

Suppose $|x(t)| \leq \psi \equiv p_2\varepsilon/2p_1(a_0+T_0)T_0$ for $t \in [s_i,s_{i+1}]$. We shall see that there is a $t_0 \in [s_i,s_{i+1}]$ such that $|x(t_0)| \geq p_2\varepsilon/4p_1$. Then we can conclude that there is a $t_0\varepsilon[s_i,s_{i+1}]$ such that

$$|x(t_0)| \geq \min \{\psi,p_2\varepsilon/4p_1\} = \gamma.$$

Suppose $(t_1,t_2) \subseteq (s_i,s_{i+1})$. Then

$$|x(t_2) - x(t_1)| = \left| \int_{t_1}^{t_2} \dot{x}(\tau)d\tau \right| = \left| \int_{t_1}^{t_2} (A(\tau)x(\tau) - B^T(\tau)y(\tau))d\tau \right|$$

$$\geq \left| \int_{t_1}^{t_2} B^T(\tau)y(\tau)d\tau \right| - a_0\psi(t_2-t_1)$$

and, if $t_0\varepsilon(s_i,s_{i+1})$, then

$$\left| \int_{t_1}^{t_2} (B^T(\tau)y(\tau) - B^T(\tau)y(t_0))d\tau \right| \leq \int_{t_1}^{t_2} |B(\tau)||y(\tau) - y(t_0)|d\tau$$

$$\leq \int_{t_1}^{t_2} |B(\tau)| \left| \int_{t_0}^{\tau} \dot{y}(\sigma)d\sigma \right| d\tau \leq \int_{t_1}^{t_2} |B(\tau)| \left(\int_{t_0}^{\tau} |B(\sigma)||x(\sigma)|d\sigma \right) d\tau$$

$$\leq \int_{t_1}^{t_2} |B(\tau)| \left(\int_{s_i}^{s_{i+1}} |B(\sigma)|\psi d\sigma \right) d\tau$$

$$\leq \psi \int_{s_i}^{s_{i+1}} |B(\sigma)|d\sigma \int_{s_i}^{s_{i+1}} |B(\tau)|d\tau$$

$$\leq \psi \quad u_{i+1}(s_{i+1}-s_i)u_{i+1}(s_{i+1}-s_i) \leq \psi \; T_0^{\;2}.$$

Thus

$$|x(t_2) - x(t_1)| \geq \left| \int_{t_1}^{t_2} B^T(\tau)y(t_0)d\tau \right| - \psi T_0^{\;2} - a_0\psi T_0.$$

Now, let $t_0 = s_i$. Given $y_0 = y(s_i)$, there is an interval $(t_1,t_2) \subseteq (s_i,s_{i+1})$ such that

$$|x(t_2) - x(t_1)| \geq \varepsilon_0 \frac{p_2}{p_1} - \psi T_0(T_0+a_0)$$

$$\geq \frac{1}{2} \varepsilon_0(p_2/p_1).$$

Therefore, there is a $t_0 \in [s_i,s_{i+1}]$ such that

$$|x(t_0)| \geq \frac{1}{4} \varepsilon_0(p_2/p_1) = p_2\varepsilon_0/4p_1.$$

This completes the proof of Theorem 4.

Proof of Theorem 5:

1. This is a corollary to Theorem 4. We need to identify $T_0,T_1,T_2,\varepsilon_0,s_i \to \infty$, and u_i so that Definition 2 is satisfied.

Since $B_0(t)$ is bounded and uniformly exciting, we have T,ε, and $r_i \to \infty$ so that Definition 1 is satisfied. Since $B_0(t)$ satisfies assumption E, we have γ,δ,L, and $t_i \to \infty$ from Definition 3.

Let $|B_0(t)| \leq b_0$ for all $t \in R^+$.

Define

$$T_0 = b_0 T_*$$

$$T_1 = 2T + 2\delta$$

$$T_2 = \delta/2$$

$$\varepsilon_0 = h_* \varepsilon/2mT$$

$$u_i = b_0 v_i$$

Let $\hat{T}$ be large enough so that if $s_j \geq \hat{T}$, then $s_{j+1} - s_j \leq \min\{\delta/4, T_0\}$. We lose no generality assuming that the initial time for solutions is greater than or equal to $\hat{T}$.

2. Now parts 1 and 2 of Definition 2 follow easily. Let us establish part 3. Suppose $I = [t_0, t_0 + T_1]$ for some $t_0 \geq \hat{T}$. Then, there is an index i_0 with $(r_{i_0}, r_{i_0+1}) \subseteq (t_0+\delta, t_0+\delta+2T)$. Let y_0 be a unit vector in R^m. By the uniform excitedness of $B_0(t)$, there is an interval $(a,b) \subseteq (r_{i_0}, r_{i_0+1})$ such that

$$\left| \int_a^b B_0^T(\tau) y_0 d\tau \right| \geq \varepsilon.$$

It follows that there is a subset Λ_0 of (a,b) of non-zero measure and a component of $B_0^T(\tau)y_0$, say the k^{th}, such that either $(B_0^T(\tau)y_0)_k \geq \varepsilon/mT$ for $\tau \in \Lambda_0$ or $(B_0^T(\tau)y_0)_k \leq -\varepsilon/mT$ for $\tau \in \Lambda_0$. For simplicity, and without loss of generality, let us assume $(B_0^T(\tau)y_0)_k \geq \varepsilon/mT$ for $\tau \in \Lambda_0$. Then there is a $t_* \varepsilon (a,b)$ such that $(B_0^T(t_*)y_0)_k \geq \varepsilon/mT$ and $t_* \neq t_i$ for all i.

Now, applying the fact that $B_0(t)$ obeys assumption E, we have $t_* \varepsilon (t_i, t_{i+1})$ for some i and there is an interval $(c,d) \subseteq (t_i, t_{i+1})$ with $t_* \varepsilon (c,d)$ and $d - c \geq \delta$ with $(B^T(t)y_0)_k \geq \varepsilon/2mT$ for $t \, \varepsilon (c,d)$. We may assume $d - c = \delta$.

Since $t_* \varepsilon (r_{i_0}, r_{i_0+1}) \subseteq (t_0+\delta, t_0+\delta+2T)$ and $t_* \varepsilon (c,d)$ with $d - c = \delta$, we have $(c,d) \subseteq (t_0, t_0+\delta+2T+\delta) = I$. Since $s_{j+1} - s_j \leq \delta/4$ for $s_j \geq t_0$, the interval $(c+\delta/4, d-\delta/4)$ can be covered by intervals (s_j, s_{j+1}) contained in (c,d). Let S denote the union of these intervals. Then $\mu(S) \geq \delta/2 = T_2$.

Suppose $(s_j, s_{j+1}) \subseteq S$. Then $(B_0^T(t)y_0)_k \geq \varepsilon/2mT$ for $t \in (s_j, s_{j+1})$, and therefore

$$\left| \int_{s_j}^{s_{j+1}} h(\tau)B_0^T(\tau)y_0 d\tau \right| \geq \left| \int_{s_j}^{s_{j+1}} h(\tau)(B_0^T(\tau)y_0)_k d\tau \right|$$

$$\geq \frac{\varepsilon}{2mT} \int_{s_j}^{s_{j+1}} |h(\tau)| d\tau \geq \frac{\varepsilon}{2mT} h_* = \varepsilon_0$$

This completes the proof of Theorem 5.

We need the following lemma for the proof of Theorem 3.

Lemma: Let $f: R^+ \to R^+$ be a smooth function obeying the following conditions:

1. $0 \leq \dot{f}(t) \leq kf(t)$
2. $0 \leq \ddot{f}(t) \leq k\dot{f}(t)$ for some constant k and all $t \in R^+$
3. $f(t) \to \infty$ as $t \to \infty$.

Let $h: R^+ \to R^1$ be piecewise continuous. Then the solution $w(t)$ to the initial value problem

$$\ddot{w} = -f(t)w + h(t) \ , \ w(t_0) = \dot{w}(t_0) = 0, \tag{4}$$

for sufficiently large t_0 satisfies the inequality

$$|w(t)| \leq 6(f(t))^{-1/4} \int_{t_0}^{t} |h(\tau)| (f(\tau))^{-1/4} d\tau.$$

Proof: Consider the equation $\ddot{u} = -f(t)u$. By the Liouville-Green approximation ([10], pp. 681-684), this equation has complex conjugate solutions:

$$u_{\pm}(t) = (f(t))^{-1/4} \exp\{\pm i \int (f(\tau))^{1/2} d\tau\} (1 + \varepsilon_{\pm}(t))$$

with the error terms $\varepsilon_+(t)$ and $\varepsilon_-(t)$ obeying the estimates

$$|\varepsilon_{\pm}(t)| \leq E(t_0) \ , \ (f(t))^{-1/2} |\dot{\varepsilon}_{\pm}(t)| \leq E(t_0)$$

where

$$E(t_0) = \exp\{k_0(f(t_0))^{-1/2}\} - 1$$

for some positive constant t_0. Note that $|(u_+(t),u_-(t))| \lesssim 2(f(t))^{-1/4}(1 + E(t_0))$.
Now

$$U(t) = \begin{bmatrix} u_+(t) & u_-(t) \\ \dot{u}_+(t) & \dot{u}_-(t) \end{bmatrix}$$

is a fundamental matrix for $\ddot{u} = -f(t)u$. Let t_0 be large enough so that $E(t) \leq \frac{1}{8}$ for $t \geq t_0$. Then $5 \geq \det(U(t)) \geq 1$ for all $t \geq t_0$.

Thus, if $w(t)$ is a solution to (4) with $w(t_0) = \dot{w}(t_0) = 0$, then

$$\begin{bmatrix} w(t) \\ \dot{w}(t) \end{bmatrix} = \int_{t_0}^{t} U(t)U^{-1}(\tau) \begin{bmatrix} 0 \\ h(\tau) \end{bmatrix} d\tau$$

and therefore

$$w(t) = \int_{t_0}^{t} \frac{h(\tau)}{\det U(\tau)} < (u_+(t),u_-(t)),\ (-u_-(\tau),u_+(\tau)) > d\tau$$

where <,> denotes the dot product. Thus

$$\begin{aligned} |w(t)| &\leq \int_{t_0}^{t} |h(\tau)|(2(f(t))^{-1/4}(1 + E(t_0)))(2(f(\tau))^{-1/4}(1 + E(t_0)))d\tau \\ &= 4(1 + E(t_0))^2(f(t))^{-1/4} \int_{t_0}^{t} |h(\tau)|(f(\tau))^{-1/4}d\tau \\ &\lesssim 6(f(t))^{-1/4} \int_{t_0}^{t} |h(\tau)|(f(\tau))^{-1/4}d\tau. \end{aligned}$$

Proof of Theorem 3. We have $\dot{x} = -x + b^T(t)y$ and $\dot{y} = b(t)x$. Let $(x(t),y(t)) \in R^{m+1}$ denote a solution to (1) with initial state $(x(t_0),y(t_0))$ obeying $x(t_0) = 0$, $\dot{x}(t_0) = 0$, $|y(t_0)| = 1$ and $|y_1(t_0)| \lesssim 1/2$. Note that

$$\ddot{x}(t) = - |b(t)|^2x(t) + h(t)$$

where $h(t) = x(t) + (b^T(t) - \dot{b}^T(t))y(t)$. Thus

$$|h(t)| \leq |x(t)| + 2mke^{(1/2-\varepsilon)t}|y(t)|$$

$$\leq 1 + 2mke^{(1/2-\varepsilon)t}.$$

Note also that $|b(t)| \leq k_0 e^t$ for some $k_0 > 0$.

Now, it follows from the lemma that

$$|x(t)| \leq 6|b(t)|^{-1/2} \int_{t_0}^{t} |h(\tau)||b(\tau)|^{-1/2} d\tau$$

$$\leq 6\, k_0^{-1/2} e^{-\frac{1}{2}t} \int_{t_0}^{t} (1 + 2mke^{(1/2-\varepsilon)\tau}) k_0^{-1/2} e^{-1/2\tau} d\tau$$

$$\leq 6\, k_0^{-1} e^{-\frac{1}{2}t} \int_{t_0}^{t} e^{-1/2\tau} + 2mke^{-\varepsilon\tau} d\tau$$

$$\leq k_1 e^{-\frac{1}{2}t} \quad \text{for some } k_1 > 0.$$

Let j be fixed with $2 \leq j \leq m$. Then $\dot{y}_j(t) = b_j(t)x(t)$, and

$$|y_j(t) - y_j(t_0)| \leq \int_{t_0}^{t} |b_j(\tau)||x(\tau)| d\tau \leq \int_{t_0}^{t} ke^{(1/2-\varepsilon)\tau} k_1 e^{-\frac{1}{2}\tau} d\tau$$

$$\leq kk_1 \int_{t_0}^{t} e^{-\varepsilon\tau} d\tau.$$

It follows that, for sufficiently large t_0, $|y_j(t) - y_j(t_0)| \leq 1/2m$ for all $t \geq t_0$ for all j. But $|y(t_0)| = 1$, so this implies some $y_j(t)$ does not converge to 0. Thus, (1) is not asymptotically stable.

Appendix on Almost Periodic Functions

In this appendix, Theorem 1 is proven. First, however, we need some general information about almost periodic functions.

Definition: The mxn matrix $B(t)$ is "uniformly almost periodic" (a.p.) if $B(t)$ is continuous and, for any $\gamma > 0$, there is an $L = L(B,\gamma) > 0$ such that in any interval of length L there is a t_* such that $|B(t+t_*) - B(t)| \leq \gamma$ for all $t \in R^1$.

This is only one of several equivalent definitions of a.p. See Hale [3], pp. 315-325, or Fink [2], Chapter 1, for basic facts about a.p. functions. We will need the following:

Let $B(t)$ denote a nxm matrix.

1. $B(t)$ is a.p. if and only if $B_{ij}(t)$ is a.p. for all i,j.
2. If $B_{ij}(t)$ is a linear combination of continuous periodic functions, then $B_{ij}(t)$ is a.p.
3. If $B(t)$ is a.p., then $B(t)$ is uniformly bounded and uniformly continuous.

Proof of Theorem 1: We state the following claim, to be proven later. Let S_0 denote the unit vectors in R^m.

Claim: The following are equivalent:

a. For each $y_0 \in S_0$, there is a $t_0 \in R^+$ such that $|B^T(t_0)y_0| \neq 0$.

b. There are positive constants γ and T such that, for each $y_0 \in S_0$, there is a $t_0 \in [0,T]$ such that $|B^T(t_0)y_0| \geq \gamma$.

c. There are positive constants γ_0 and T_0 such that, for each $y_0 \in S_0$ and $t_1 \in R^+$, there is a $t_2 \in [t_1,t_1 + T_0]$ such that $|B^T(t_2)y_0| \geq \gamma_0$.

Now, using the claim, let us complete the proof of Theorem 1. By hypothesis, part a of the claim holds. Let γ_0 and T_0 be as in part c. Let $y_0 \in S_0$ and $t_1 \in R^+$. Then there is a $t_2 \in R^+$ such that $|B^T(t_2)y_0| \geq \gamma_0$ for some $t_2 \in [t_1,t_1 + T_0]$. Therefore, there is a component of $B^T(t_2)y_0$, say the k^{th}, such that either $(B^T(t_2)y_0)_k \geq \gamma_0/m$ or $(B^T(t_2)y_0) \leq -\gamma_0/m$. Without loss of generality, assume

$(B^T(t_2)y_0)_k \geq \gamma_0/m$. Now, by the uniform continuity of $B(t)$, there is a $\delta > 0$ such that if $|t - s| < \delta$, then $|B(t) - B(s)| \leq \gamma_0/2m$. Thus $|(B^T(t)y_0)_k - (B^T(t_2)y_0)_k| \leq \gamma_0/2m$ for $|t - t_2| < \delta$. Thus there is an interval $(a,b) \subseteq (t_1, t_1 + T_0)$ with $b - a \geq \delta_0 \equiv \min\{\delta, T_0\}$ such that

$$\left| \int_a^b B^T(\tau)y_0 d\tau \right| \geq \left| \int_a^b (B^T(\tau)y_0)_k d\tau \right| \geq \frac{\gamma_0(b-a)}{2m} \geq \frac{\gamma_0\delta_0}{2m} .$$

This proves that $B(t)$ is uniformly exciting.

Proof of the Claim: Assume a. is true. Let $y_0 \in S_0$. There is a $t_0 \in R^+$ such that $|B^T(t_0)y_0| = 2\gamma_{y_0} > 0$. Now, there is a ball $S(y_0)$ about y_0 in R^m such that if $y \in S(y_0)$, then $|B^T(t)y| \geq \gamma_{y_0}$. Thus, the balls $S(y_0)$ for $y_0 \in S_0$ cover S_0, so there is a finite subcover, $\{S(y_1),\ldots,S(y_r)\}$ with associated $t_1,\ldots,t_r$. Let $\gamma = \min\{\gamma_{y_1},\ldots,\gamma_{y_r}\} > 0$ and $T = \max\{t_1,\ldots,t_r\}$. Then, if $y_0 \in S_0$, there is a $t_0 \in \{t_1,\ldots,t_r\} \subseteq [0,T]$ such that $|B^T(t_0)y_0| \geq \gamma$. This proves b.

Assume b. is true. Let $y_0 \in S_0$ and $t_1 \in R^+$. There is a $t_0 \in [0,T]$ such that $|B^T(t_0)y_0| \geq \gamma$. Let $L = L(B,\gamma/4)$ and choose $t_* \in [t_1-t_0, t_1-t_0+L]$ as in the definition of a.p. Then $|B^T(t_0+t_*) - B^T(t_0)| \leq \gamma/4$, which implies $|B^T(t_0+t_*)y_0| \geq 3\gamma/4$. Since $t_0 + t_* \in [t_1, t_1+L]$ we have proven c.

That c. implies a. is trivial.

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